

INDEPENDENCE OF ITERATED WHITEHEAD DOUBLES

JUANITA PINZÓN-CAICEDO

(Communicated by David Futer)

ABSTRACT. A theorem of Furuta and Fintushel-Stern provides a criterion for a collection of Seifert fibered homology spheres to be independent in the homology cobordism group of oriented homology 3-spheres. In this article we use these results and some 4-dimensional constructions to produce infinite families of positive torus knots whose iterated Whitehead doubles are independent in the smooth concordance group.

1. INTRODUCTION

A smooth knot is a smooth embedding of the circle S^1 into the 3-sphere S^3 , and as a consequence of the (unpublished) work of Thurston, every knot is either hyperbolic, a torus, or a satellite knot [28]. Hyperbolic knots are those whose complement admits a hyperbolic structure, torus knots are those that lie on the surface of an unknotted torus in S^3 and are specified by a pair of coprime integers p and q , and finally, satellite knots are those whose complement contains an incompressible, non-boundary-parallel torus. Moreover, satellite knots are constructed from two given knots, P and K , in the following way. Let $P \sqcup J$ be a link in S^3 with J an unknot so that P lies in the solid torus $V = S^3 \setminus N(J)$. The satellite knot with pattern $P \sqcup J$ and companion K is denoted by $P(K)$ and is obtained as the image of P under the embedding of V in S^3 that knots V as a tubular neighborhood of K , using the 0-framing of K .

The set of isotopy classes of knots is a semigroup with connect-sum as its binary operation [20, Chapter 1, Section 5]. To obtain a group structure topologists consider another equivalence relation on the set of knots, concordance. Two smooth knots $K_0, K_1 \subset S^3$ are smoothly concordant if there exists a smoothly and properly embedded annulus A into the cylinder $I \times S^3$ that restricts to the given knots at each end. However, if the embedding of A into $I \times S^3$ is locally flat and proper, the knots K_0 and K_1 are said to be topologically concordant. These two different approaches give rise to the two related theories of smooth and topological concordance, and both induce an abelian group structure on the set of knots with connected sum as the operation. Studying the relationship between smooth and topological concordance is an area of active research in knot theory especially because the difference between smooth and topologically slice knots is related to subtle differences in the set of differentiable structures on 4-manifolds. One approach to this problem is to understand the group structure of the “forgetful homomorphism” $\mathcal{C} \rightarrow \mathcal{C}_{\text{TOP}}$. The 1980’s saw the birth of tremendously important results in that direction: those of

Received by the editors January 9, 2018, and, in revised form, May 22, 2018.

2010 *Mathematics Subject Classification.* Primary 57M25, 57M27, 57N70, 57Q60.

©2018 American Mathematical Society

Freedman [10–12] and Donaldson [4–6]. These results allowed topologists to actually understand just how vastly different \mathcal{C} and \mathcal{C}_{TOP} are. Indeed, on the one hand, Freedman’s theorem implies that knots with Alexander polynomial 1 are topologically slice, and on the other, Donaldson’s theorem can be used to show that some knots with Alexander polynomial 1 are not smoothly slice. As an example, if the link $D^r \sqcup J$ is the r -th iterated Whitehead link, $D^r(K)$ is called the (untwisted positively clasped) r -th iterated Whitehead double of K . Since the pattern D^r is an unknot in S^3 and is trivial in $H_1(V; \mathbb{Z})$, the satellite $D^r(K)$ has trivial Alexander polynomial and is thus topologically slice. The main result of this article is the following:

Theorem. *If a sequence $\{(p_i, q_i)\}_i$ of pairs of relatively prime and positive integers satisfies that $p_i q_i (4p_i q_i - 1) < p_{i+1} q_{i+1} (p_{i+1} q_{i+1} - 1)$, then no non-trivial linear combination of elements of the set $\{D^{r_i}(T_{p_i, q_i})\}_{i=1}^\infty$ is smoothly slice.*

Part of the motivation to study independence of Whitehead doubles comes from the following problem in Kirby’s list [7].

Problem 1 (1.38 in [7]). The untwisted double of a knot is slice if and only if the knot is slice.

It was Akbulut [1] [2, Exercise 11.4, p. 143] who first used Donaldson’s theorem to prove that $D(T_{2,3})$, the Whitehead double of the right handed trefoil knot, is not smoothly slice. Akbulut’s technique was extended by Cochran-Gompf [3] to show that Whitehead doubles of positive torus knots have infinite order. The introduction of Seiberg-Witten invariants and their relationship with the invariant of Thurston-Benequin allowed Rudolph [24, 25] to show that iterated doubles of strongly quasipositive knots (of which torus knots are an example) are not smoothly slice. Next, the work of Ozsváth and Szabo in Heegaard Floer was used by Hedden [15] to generalize the result of Rudolph, and by Park [22] to show that $D(T_{2,2m+1})$ and $D^2(T_{2,2m+1})$ generate a \mathbb{Z}^2 -summand of \mathcal{C} . Finally, the work of Furuta [13] and Fintushel-Stern [9] on the theory of instantons and Chern-Simons invariants was used first by Endo [8] to show that the collection of pretzel knots $\mathcal{P} = \{K(p, q, r) \mid (p, q, r) \text{ odd integers satisfying } pq + qr + rp = -1 \text{ and } p, q, r \neq \pm 1\}$ generates an infinite rank subgroup of the kernel of $\mathcal{C} \rightarrow \mathcal{C}_{\text{TOP}}$, and later by Hedden-Kirk [16] to show that certain families of Whitehead doubles of positive torus knots generate a subgroup of \mathcal{C} of maximal infinite rank. Their work was generalized to other Whitehead-like patterns in [23], and the present work can be regarded as its sequel.

As usual with concordance, the methods involve the topology of 3- and 4-manifolds. If a knot K is slice, then on one hand, the 2-fold cover of S^3 branched over K bounds a smooth 4-manifold with the same $\mathbb{Z}/2$ homology as the 4-ball, and on the other, surgery on S^3 along K bounds a 4-manifold with an embedded 2-sphere. Then, obstructions to these 3-manifolds from bounding smooth 4-manifolds with the prescribed topology provide obstructions to the knots used in their constructions from being slice. In the present article the obstruction to the sliceness of sums of iterated doubles will come from the study of the moduli space of ASD connections on a 4-manifold with cylindrical ends modeled over the 2-fold covers of S^3 branched over the doubles.

Outline. To establish the result we will translate the question of concordance of knots into a question of homology cobordism of their 2-fold covers, and so Section

2 focuses on a description of the topology of the covers of iterated Whitehead doubles. The basics of the results of Furuta [13] and Fintushel-Stern [9] regarding independence in $\Theta_{\mathbb{Z}/2}^3$ will be briefly presented in Section 3. However, this result requires a particular geometric structure on 3-manifolds which 2-fold covers over iterated doubles lack. This issue can be sidestepped via cobordisms, and so Section 4 describes constructions of definite cobordisms from our 3-manifolds to 3-manifolds with the right geometry. With all the ingredients at hand, Section 5 presents the final proof.

2. SATELLITES AND THEIR COVER

The classification of knots up to concordance, and in particular the study of sliceness, is a problem that involves both 3- and 4-manifold topology. Double branched covers, for example, provide a connection between problems in knot theory and other questions in low-dimensional topology. To be more specific, obstructions to sliceness can be found using homology cobordism, a 3-dimensional analogue to concordance. Call two oriented $\mathbb{Z}/2$ -homology spheres Σ_0 and Σ_1 homology cobordant if there is an oriented smooth 4-manifold C with oriented boundary $-\Sigma_0 \sqcup \Sigma_1$ and such that $H_*(C; \mathbb{Z}/2) = H_*(I \times S^3; \mathbb{Z}/2)$. In this case, we call C a $\mathbb{Z}/2$ -homology cobordism from Σ_0 to Σ_1 . The set of $\mathbb{Z}/2$ -homology cobordism classes of $\mathbb{Z}/2$ -homology spheres forms an abelian group denoted by $\Theta_{\mathbb{Z}/2}^3$ and with connected sum as the group operation. To establish the relationship between $\mathbb{Z}/2$ -homology cobordism and concordance, for a knot K in S^3 , denote by $\Sigma(K)$ the 2-fold cover of S^3 branched over K . Notice that if two knots K_0 and K_1 are concordant via an annulus A , then the double cover of the cylinder $I \times S^3$ branched along A is a $\mathbb{Z}/2$ -homology cobordism between the 3-manifolds $\Sigma(K_0)$ and $\Sigma(K_1)$ and so there is a well defined assignment $K \rightarrow \Sigma(K)$ from the smooth concordance group \mathcal{C} into $\Theta_{\mathbb{Z}/2}^3$. Moreover, since the separating S^2 that appears in a connected sum $K_0 \# K_1$ lifts to a separating sphere of the 2-fold cover of the sum, this map is a group homomorphism. Thus, the question of the existence (or lack thereof) of a slicing disk for a knot K translates into the question of the existence of a 4-manifold with the same $\mathbb{Z}/2$ -homology groups as the 4-ball and with boundary $\Sigma(K)$. This section details a special and particularly useful decomposition of branched covers of satellites.

First, recall that the untwisted satellite knot $P(K)$ with pattern P and companion K is obtained as the image of P under the embedding of an unknotted solid torus V in S^3 containing P that knots V as a tubular neighborhood of K , using the 0-framing of K . For n any integer, an n -twisted satellite is obtained using the n -framing of K in the identification. Whitehead doubles are an important example of untwisted satellites and are obtained by using the Whitehead link as the pattern of the satellite operation. Similar examples arise by considering Whitehead doubles of other Whitehead doubles. These examples are called iterated Whitehead doubles. The images to the right of Figure 2 show the patterns of the first three iterations of this satellite operation.

Now, regarding general covers of S^3 branched over satellites, denote by $\Sigma_q(P(K))$ the q -fold branched cover of S^3 branched over $P(K)$. If J is a parallel copy of a meridional curve for the solid torus V so that V can be identified with $S^3 \setminus N(J)$, then J has l lifts into $\Sigma_q(P)$, where $l = \gcd(q, lk(J, P))$. It follows that $\Sigma_q(P(K))$, the q -fold branched cover of S^3 branched over $P(K)$, is formed from

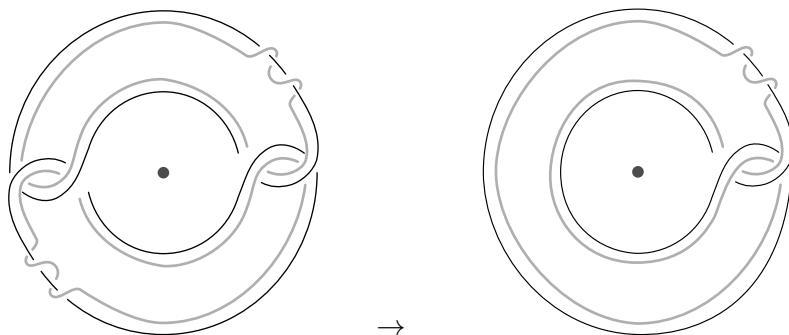


FIGURE 1. The cover $\Sigma(D^1) \rightarrow S^3$, the knot J and its longitude, as well as their lifts.

$\Sigma_q(P(U))$ (where U is the unknot) by removing neighborhoods of the lifts of J and replacing each with the q/l -cyclic cover of the complement of K . See [18] and [27] for the details. For Whitehead doubles and $q = 2$ specifically, we have the following decomposition; the details can be found in [16] or [23]:

$$(1) \quad \Sigma(D(K)) = S^3 \setminus N(T_{2,4}) \underset{\tilde{\phi}}{\cup} 2E(K).$$

Here $E(K)$ denotes the knot exterior $S^3 \setminus N(K)$, and the gluing map $\tilde{\phi}$ identifies each copy of the meridian μ_K of the knot K to the curve $(-2, 1)$ in each component of $\partial N(T_{2,4})$. See Figure 1 for a visualization of this cover. This description is obtained by identifying the covering space of $S^3 \setminus N(D)$ with $S^3 \setminus N(T_{2,4})$, where $T_{2,4}$ is the (unoriented) two component torus link determined by $(2, 4)$. The following proposition relies on this identification to describe a similar decomposition for the double covers $\Sigma(D^r(K))$ for $r > 1$.

Proposition 1. *Given a knot $K \subseteq S^3$ and an integer $r \geq 1$, let $D^r(K)$ be the r -th iterated Whitehead double of K . The 2-fold cover $\Sigma(D^r(K))$ of S^3 branched over $D^r(K)$ has a decomposition*

$$\Sigma(D^r(K)) = S^3 \setminus N(D_{-2}^{r-1}(T_{2,4})) \underset{\tilde{\phi}}{\cup} 2E(K),$$

where $2E(K)$ denotes two disjoint copies of the knot exterior $S^3 \setminus N(K)$ and $D_{-2}^{r-1}(T_{2,4})$ denotes the iterated double of the link $T_{2,4}$ twisted by -2 . Additionally, the gluing map $\tilde{\phi}$ identifies each copy of the meridian μ_K of the knot K with the longitude of one of the components of $D_{-2}^{r-1}(T_{2,4})$.

Proof. Denote by $\Sigma(V, D^r)$ the 2-fold cover of the solid torus $V = S^3 \setminus N(J)$ branched over the pattern D^r . Results from [18] and [27] show that the 2-fold cover of S^3 branched over $D^r(K)$ is completely determined by $\Sigma(V, D^r)$, the homology class of the two lifts of $\mu_V = \lambda_J$ to $\Sigma(V, D^r)$, and the knot exterior $S^3 \setminus N(K)$. Since the iterated Whitehead link $D^r \sqcup J$ is symmetric, $\Sigma(V, D^r)$ is diffeomorphic to the 2-fold cover of $S^3 \setminus N(D^r)$ branched over J . The latter space can be obtained from $\Sigma(S^3, J) \cong S^3$ after removing a tubular neighborhood of the lift of D^r , and so a description of the lift of D^r and the lift of its longitude will be enough to completely understand $\Sigma(D^r(K))$. With that in mind, consider the untwisted satellite map

$\psi : V \rightarrow N(D)$ with pattern D^{r-1} and realize $D^r \subseteq N(D)$ as $D^r = D^{r-1}(D) = \psi(D)$. Next, related to the identification included as (1), notice that if $p : \Sigma(S^3, J) \rightarrow S^3$ is the cover map, then the restriction of p to $N(T_{2,4}) = p^{-1}(N(D))$ is a cyclic covering space $p : N(T_{2,4}) \rightarrow N(D)$, and so basic covering space theory [14, Proposition 1.33] shows that corresponding to the map $\psi : V \rightarrow N(D)$, and for each choice of component A_i of $T_{2,4}$, there exists a homeomorphism $\tilde{\psi}_i : V \rightarrow N(A_i)$ satisfying $p \circ \tilde{\psi}_i = \psi$. Moreover, $\tilde{\psi}_i$ carries μ_V to μ_{A_i} , and λ_V to $-2\mu_{A_i} + \lambda_{A_i}$ and therefore the lift of D^r to $N(T_{2,4})$ is given by $\tilde{\psi}_1(D^{r-1}) \sqcup \tilde{\psi}_2(D^{r-1}) = D_{-2}^{r-1}(T_{2,4})$, the -2 -twisted $r-1$ iterated double of the link $T_{2,4}$. Finally, since each $\tilde{\psi}_i$ is a twisted satellite map in its own right, they identify the longitude of D^{r-1} with a longitude of its image $D_{-2}^{r-1}(A_i)$, thus showing that the lifts of a longitude of D^r to $N(T_{2,4}) \subset \Sigma(S^3, J)$ are precisely the longitudes of the components of $D_{-2}^{r-1}(T_{2,4})$. \square

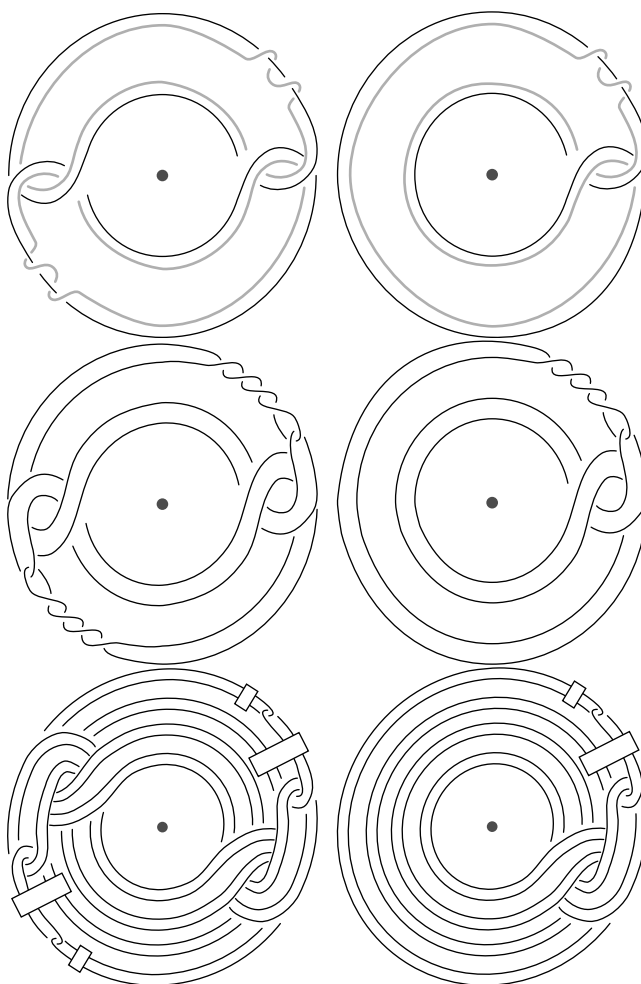


FIGURE 2. The cover $\Sigma(D^r) \rightarrow S^3$ for $r = 1, 2, 3$.

3. THE 4-DIMENSIONAL OBSTRUCTION

Since for every $r \geq 1$ the knot D^r is trivial in S^3 and has linking number 0 with a meridional disk of the unknotted solid torus containing it, every iterated untwisted Whitehead double has trivial Alexander polynomial, and so Freedman's theorem [10–12] implies that every iterated untwisted Whitehead double is topologically slice. As a consequence, classical invariants do not detect information about their smooth concordance type, and so smooth techniques like gauge theory are necessary to obtain that information. In this article we will use the internal structure of the moduli space of anti-self dual connections on a Seifert fibration with three exceptional fibers to obtain an obstruction to the sliceness of sums of iterated Whitehead doubles of torus knots. Similar to the way in which Donaldson's theorem applies instantons to describe the specific form the intersection form of a closed and definite 4-manifold takes, the theory of instantons can be applied to manifolds with boundary (via the addition of cylindrical ends) to obstruct the existence of certain 4-manifolds. This theory was first developed by Furuta [13] and Fintushel and Stern [9], and extended by Hedden and Kirk [16]. For the technical details we refer the reader to their articles. The following theorem is a restatement of their combined results that is free of the technicalities of gauge theory. To establish notation, given (p, q, s) a set of relatively prime and positive integers, let $\Sigma(p, q, s)$ be the Seifert fibered homology sphere with exceptional fibers of orders p, q, s .

Theorem 2. *Let p_i, q_i be relatively prime integers and let k_i be a positive integer for $i = 1, \dots, N$. If $\{\Sigma_i\}_{i=1}^N$ is a family of Seifert fibered homology 3-spheres such that $\Sigma_i = \Sigma(p_i, q_i, k_i p_i q_i - 1)$ and satisfying*

$$(2) \quad p_i q_i (k_i p_i q_i - 1) < p_{i+1} q_{i+1} (k_{i+1} p_{i+1} q_{i+1} - 1),$$

then no linear combination of elements in $\{\Sigma_i\}_{i=1}^N$ cobounds a smooth 4-manifold X with negative definite intersection form and such that $H_1(X; \mathbb{Z}/2) = 0$.

The result is obtained after examining the moduli space \mathcal{M} of ASD connections on an $\mathrm{SO}(3)$ bundle over a 4-manifold X with boundary a sum of elements of $\{\Sigma_i\}_{i=1}^N$, and determined by the fibration $\Sigma(p_N, q_N, k_N p_N q_N - 1) \rightarrow S^2$. The (virtual) dimension of \mathcal{M} can be calculated using the Neumann-Zagier formula [21] and can be shown to be exactly 1 when p_N and q_N are relatively prime positive integers, and $k_N \geq 1$. In addition, \mathcal{M} can be shown to be a 1-dimensional manifold, and so its singularities can be regarded as boundary points. These singularities of \mathcal{M} can be shown to correspond to reducible connections, the number of which can be computed in terms of the order of the torsion subgroup of $H_1(X; \mathbb{Z})$ and the number of even factors in it. Finally, compactness of \mathcal{M} is guaranteed after requiring $1/4 < p_{i+1} q_{i+1} (k_{i+1} p_{i+1} q_{i+1} - 1)$ to rule out bubbling, and $p_i q_i (k_i p_i q_i - 1) < p_{i+1} q_{i+1} (k_{i+1} p_{i+1} q_{i+1} - 1)$ to rule out leaking. If the 4-manifold X were negative definite and had $H_1(X; \mathbb{Z}/2) = 0$, then \mathcal{M} would be a compact 1-dimensional space with an odd number of boundary components, which is a contradiction, and so at least one of the hypotheses about the topology of X had to be incorrect.

This purely 4-dimensional theorem can be used to obtain an obstruction to sliceness as follows:

Remark 3 (Slice obstruction). Suppose a certain sum of knots is slice; then the sum of the double covers bounds a smooth 4-manifold Q with $\mathbb{Z}/2$ homology isomorphic to the $\mathbb{Z}/2$ homology of B^4 . However, if the double covers are cobordant to Seifert

fibered spheres via cobordisms satisfying the hypothesis of Theorem 2, then the manifold X obtained as the union of Q and the cobordisms would be a negative definite manifold with $b_1 = 0$. The existence of X contradicts Theorem 2, and so the original hypothesis of the sliceness of the sums was incorrect.

The next section describes some special constructions of cobordisms from double covers of iterated doubles to Seifert fibered spaces.

4. COBORDISMS

Recall that two oriented 3-manifolds M_0 and M_1 are said to be cobordant if there is an oriented smooth 4-manifold W with oriented boundary $-M_0 \sqcup M_1$. In this case, we call W a cobordism from M_0 to M_1 , and we call it a negative or positive definite cobordism if its intersection form is respectively negative definite or positive definite. Notice that since the covers $\Sigma(D^r(K))$ are not Seifert fibered spaces, the results explained in Section 3 cannot be applied directly to their sums. However, the 4-dimensional obstruction allows for $b_2 > 0$ as long as the 4-manifold is negative definite. In this section we will take advantage of this fact and construct definite cobordisms from the covers to some Seifert fibered spaces. It is worth mentioning that positive definite cobordisms are necessary if we want to consider sums of iterated Whitehead doubles that include negative coefficients. In fact, Donaldson's theorem and its treatment in [3] show that sums of iterated doubles with only positive coefficients are never slice.

Lemma 4. *Let $r > 1$. If K admits a series of positive-to-negative crossing changes that transform it into an unknot, then there exists a negative definite cobordism from $\Sigma(D^r(K))$ to $\text{Splice}(D_{-2}^{r-1}(U), K)$.*

Proof. By hypothesis K admits a series of positive-to-negative crossing changes that transform it into an unknot. That is, there exists a sequence of positive-to-negative crossing changes such that the i -th crossing change is obtained by performing -1 surgery on S^3 along a trivial knot γ_i that lies in $E(K)$, encloses the crossing, and has linking number 0 with the knot K . Next, consider the description of the double cover described in Theorem 1, and notice that γ_i is contained in $E(K)$ and that the gluing map identifies μ_K with the longitude of one of the components of $D_{-2}^{r-1}(T_{2,4})$. Thus, γ_i can be regarded as a subset of $\Sigma = \Sigma(D^r(K))$ with framing precisely given by its framing in S^3 . Form a 4-manifold Z by attaching 2-handles to $I \times \Sigma$ along the framed circles γ_i .

First, notice that the oriented boundary of Z consists of the disjoint union of $-\Sigma$ and the result of surgery on Σ along the knots γ_i with framing number -1 . Denote by Y the latter manifold and notice that as a consequence of Theorem 1, Y can be seen to split as the union of $E(K) \cup_{\varphi_1} (S^3 \setminus N(D_{-2}^r(T_{2,4})))$ and the result of -1 -surgery on $E(K)$ along the γ_i 's. Since the γ_i 's were chosen to give an unknotting sequence for K , -1 -surgery on $E(K)$ along the γ_i 's is isomorphic to the unknot complement and therefore isomorphic to a standard solid torus $D^2 \times S^1$. Furthermore, the isomorphism preserves meridian-longitude pairs, and thus the Seifert longitude of K gets sent to a meridional curve $\partial D^2 \times \{\text{pt.}\}$ of $D^2 \times S^1$, and the meridian of K gets sent to the longitudinal curve $\{\text{pt.}\} \times S^1$ of the solid torus $D^2 \times S^1$. Then, there is an isomorphism

$$Y \cong (S^3 \setminus N(K)) \cup_{\varphi_1} (S^3 \setminus N(D_{-2}^r(T_{2,4}))) \cup_{\varphi'_2} D^2 \times S^1,$$

where φ'_2 identifies $\partial D^2 \times \{\text{pt.}\}$ with a longitude of one of the components of $D^r_{-2}(T_{2,4})$. Specifically, if A_1, A_2 are the components of the link $T_{2,4}$ and μ_2, λ_2 are, respectively, a meridian and longitude of $D^r_{-2}(A_2)$, then the gluing map φ'_2 satisfies

$$(\varphi'_2)_*([S^1]) = (\varphi_2)_*(\mu_K) = \lambda_2 \quad \text{and} \quad (\varphi'_2)_*([\partial D^2]) = (\varphi_2)_*(\lambda_K) = \mu_2.$$

Therefore φ'_2 extends to the interior of $D^2 \times S^1$ and we have

$$Y \cong (S^3 \setminus N(K)) \cup_{\varphi_1} (S^3 \setminus N(D^{r-1}_{-2}(A_1))),$$

with

$$(\varphi_1)_*(\mu_K) = \lambda_1 \quad \text{and} \quad (\varphi_1)_*(\lambda_K) = \mu_1.$$

This shows that Y is diffeomorphic to $\text{Splice}(K, D^{r-1}_{-2}(U))$.

Next, to see that Z is negative definite, notice that since Σ is a homology sphere, the second homology group $H_2(Z; \mathbb{Z})$ admits a basis determined by the 2–handles, and the matrix representation of the intersection form of Z in terms of this basis is given by the linking matrix of the γ_i 's. It should be clear that if c is the number of crossing changes in the unknotting sequence for K , this matrix is $-I_c$, where I_c is the $c \times c$ identity matrix. \square

Lemma 5. *Let $\text{Splice}(K_0, K_1)$ denote the splice of two knots $K_0, K_1 \subseteq S^3$. There exists a negative definite cobordism from $\text{Splice}(K_0, K_1)$ to $S^3_{+1}(K_0) \# S^3_{+1}(K_1)$.*

Proof. First, consider surgery descriptions for K_0 and K_1 consisting of a link $L_i \subset S^3$ ($i = 0, 1$) whose first component represents K_i and the remaining components are unknotted circles in S^3 with zero linking number with K_i and with framing ± 1 . Then $\text{Splice}(K_0, K_1)$ has surgery diagram given by linking the first component of L_0 with the first component of L_1 in a way reminiscent of the linking of the components of the positive Hopf link. See [26, Figure 1.4, p. 9].

Next, unlink the sublinks L_0 and L_1 to produce the cobordism. Specifically, consider γ an unknotted curve in S^3 that links the first component of each L_0 and L_1 exactly once. The formula found in [17, Lemma 1.2] shows that if γ' is a curve in the boundary of a tubular neighborhood of γ that represents the homology class $(m, 1)$, then the linking number of γ and γ' in $\text{Splice}(K_0, K_1)$ is given by

$$lk(\gamma, \gamma'; \text{Splice}(K_0, K_1)) = lk(\gamma, \gamma'; S^3) - 2 = m - 2.$$

This shows that the 4–manifold obtained by attaching a 2–handle to $I \times \text{Splice}(K_0, K_1)$ along γ with framing m is negative definite as long as $m \leq 1$. Thus, if we choose $m = -1$, Kirby calculus shows that -1 surgery on $\text{Splice}(K_0, K_1)$ along γ unlinks L_0 and L_1 and changes the framing of the first component of each L_0 and L_1 from 0 to 1. It is easy to see that L_i with this new framing is a surgery description of $S^3_{+1}(K_i)$. In addition, since L_0 and L_1 are now unlinked, there is a separating 2–sphere, and thus -1 surgery on $\text{Splice}(K_0, K_1)$ along γ is diffeomorphic to $S^3_{+1}(K_0) \# S^3_{+1}(K_1)$. \square

Theorem 6. *Let K be any knot and let $\Sigma(D^r(K))$ be the 2–fold cover of S^3 branched over $D^r(K)$. For p, q a pair of integers that are relatively prime and positive, denote by $T_{p,q}$ the torus knot determined by p and q . There exist 4–manifolds $Z_{p,q}$, $P_{p,q}$, and $R_{p,q}$ such that*

- (a) $Z_{p,q}$ is a negative definite cobordism from $\Sigma(D^r(T_{p,q}))$ to $-\Sigma(p, q, pq - 1)$,

- (b) $P_{p,q}$ is a positive definite cobordism from $\Sigma(D^r(T_{p,q}))$ to $-\Sigma(p, q, 4pq - 1) \# -\Sigma(p, q, 4pq - 1)$, and
- (c) $R_{p,q}$ is a negative definite manifold with oriented boundary $-\Sigma(D^r(T_{p,q}))$.

Proof. To start, notice that Theorem 6(c) follows immediately from [23, Theorem 5.3(b)]. Similarly, the case $r = 1$ in Theorem 6(a) follows from [23, Theorem 5.3(a)] and the fact that -1 -surgery on $S^3_{1/2}(T_{p,q})$ along a longitude of $T_{p,q}$ gives $S^3_{+1}(T_{p,q})$ through a negative definite cobordism as in [3, Lemma 2.11].

Next, to obtain $Z_{p,q}$ as in Theorem 6(a), notice that for $r > 1$ Lemmas 4 and 5 give a negative definite cobordism W from $\Sigma(D^r(T_{p,q}))$ to $S^3_{+1}(D^{r-1}_{-2}(U)) \# S^3_{+1}(T_{p,q})$. Consider a curve γ in S^3 enclosing the crossings in the smallest clasp of $D^{r-1}_{-2}(U)$ and with linking number zero with the knot $D^{r-1}_{-2}(U)$. Then -1 surgery on S^3 along γ unknots $D^{r-1}_{-2}(U)$. Next, since $lk(\gamma, D^{r-1}_{-2}(U); S^3) = 0$, [17, Lemma 1.2] shows that the framing number of γ as a knot in $S^3_{+1}(D^{r-1}_{-2}(U))$ is also -1 and so the manifold $Z_{p,q}$ obtained from attaching a 2 -handle to W along γ is a negative definite cobordism from $\Sigma(D^r(T_{p,q}))$ to $S^3_{+1}(T_{p,q})$. A theorem of Moser [19] shows that $S^3_{+1}(T_{p,q}) = -\Sigma(p, q, pq - 1)$.

Lastly, to obtain the cobordism $P_{p,q}$ from Theorem 6(b), consider $+1$ -framed unknotted curves δ_1, δ_2 in S^3 that enclose the smallest clasp in each of the components of the link $D^{r-1}_{-2}(T_{2,4})$ with linking number 2 and framing $+1$. Let D_0 be the restriction to $S^3 \setminus N(D^{r-1}_{-2}(T_{2,4}))$ of a spanning disk for δ_i . Then, since φ_i identifies the meridian of each component of $D^{r-1}_{-2}(T_{2,4})$ with a longitude of $T_{p,q}$, the union of D_0 with two copies of a Seifert surface for $T_{p,q}$ in $E(T_{p,q})_i$ is a Seifert surface for δ_i in $\Sigma(D^r(T_{p,q}))$, and so the framing number for δ_i in $\Sigma(D^r(T_{p,q}))$ equals its framing number in S^3 . This shows that the 4 -manifold obtained by attaching 2 -handles to $I \times \Sigma(D^r(T_{p,q}))$ along δ_1, δ_2 in $\{1\} \times \Sigma(D^r(T_{p,q}))$ is a positive definite cobordism from $\Sigma(D^r(T_{p,q}))$ to $(S^3 \setminus N(U_2)) \cup 2(S^3 \setminus N(T_{p,q}))$, where U_2 is the 2 -component trivial link with framing -4 . This last 3 -manifold can be seen to be $S^3_{1/4}(T_{p,q}) \# S^3_{1/4}(T_{p,q})$, and the same theorem of Moser mentioned before ([19]) shows that $S^3_{1/4}(T_{p,q}) = -\Sigma(p, q, 4pq - 1)$. \square

Notice that Theorem 6(a) also exists for any knot K that admits a series of positive-to-negative crossing changes that transform it into a torus knot $T_{p,q}$ with $p, q > 0$ and that Theorem 6(c) exists for any knot K .

Lemma 7 (Lemma 2.10 from [3]). *If J admits a series of positive-to-negative crossing changes that transform it into K , then for any rational number q there exists a negative definite cobordism from $S^3_q(J)$ to $S^3_q(K)$.*

Proof. Since J admits a series of positive-to-negative crossing changes that transform it into K , there exists a link $L \subseteq S^3 \setminus N(J)$ with linking number 0 with J and such that each component of L encloses one of the crossings of J that will be changed. Also, each component of L has framing -1 . Then, the 4 -manifold W obtained by attaching 2 -handles to $I \times S^3$ along the framed link L is negative definite, and the annulus A equal to the inclusion $I \times J \rightarrow I \times S^3$ regarded as a submanifold of W restricts to J and K at each end of W and has trivial normal bundle in W . Denote by $W_q(A)$ the manifold obtained by removing $A \times D^2$ from W and replacing it with $I \times S^1 \times D^2$ using a homeomorphism which is a product on the I factor and restricts on ∂I to the q surgeries on ∂W . Then $H_2(W_q(A)) \cong H_2(W \setminus A) \cong H_2(W)$,

and hence both 4-manifolds have equivalent intersection forms. Thus $W_q(A)$ is a negative definite cobordism from $S_q^3(J)$ to $S_q^3(K)$. \square

5. INDEPENDENCE

This section is the culmination of all the work. Here we mix all the ingredients that we developed in the previous sections to finally obtain the main result as a corollary to the following 3-dimensional theorem.

Theorem 8. *Let $\{(p_i, q_i)\}_i$ be a sequence of relatively prime positive integers and let r_i be a positive integer for every i . If*

$$p_i q_i (4p_i q_i - 1) < p_{i+1} q_{i+1} (p_{i+1} q_{i+1} - 1),$$

then the family $\mathcal{F} = \{\Sigma(D^{r_i}(T_{p_i, q_i}))\}_{i=1}^\infty$ is independent in $\Theta_{\mathbb{Z}/2}^3$.

Proof. Denote by $[Y]$ the homology cobordism class of the $\mathbb{Z}/2$ -homology sphere Y and suppose by contradiction that there exist integral coefficients $c_1, \dots, c_N \in \mathbb{Z}$ such that

$$\sum_{i=1}^N c_i [\Sigma(D^{r_i}(K_i))] = 0$$

in $\Theta_{\mathbb{Z}/2}^3$. The supposition implies the existence of an oriented 4-manifold Q with the $\mathbb{Z}/2$ homology of a punctured 4-ball and with boundary

$$\partial Q = \#_{i=1}^N \left(\#_{j=1}^{c_i} \Sigma(D^{r_i}(K_i)) \right).$$

Attaching 3-handles to Q we can further assume that

$$\partial Q = \bigsqcup_{i=1}^N c_i \Sigma(D^{r_i}(K_i)).$$

Here we use cY to denote the disjoint union of c copies of Y if $c > 0$ and $-c$ copies of $-Y$ if $c < 0$. In addition, and without loss of generality, further assume that $c_N \geq 1$. Augment Q using the cobordisms constructed in Theorem 6, namely, let

$$X = Q \cup \left(Z_{p_N, q_N} \right) \cup \left(\bigsqcup_{c_i > 0} R_{p_i, q_i} \right) \cup \left(\bigsqcup_{c_i < 0} -P_{p_i, q_i} \right).$$

Thus, X is a negative definite 4-manifold with oriented boundary

$$\partial X = -\Sigma(p_N, q_N, p_N q_N - 1) \sqcup \left(\bigsqcup_{c_i < 0} \Sigma(p_i, q_i, 4p_i q_i - 1) \right).$$

Additionally, since the first $\mathbb{Z}/2$ -homology groups of Z_{p_N, q_N} , $-P_{p_i, q_i}$, R_{p_i, q_i} , and Q are trivial, the Mayer-Vietoris theorem shows that $H_1(X, \mathbb{Z}/2) = 0$. This would imply that the Seifert fibered spaces $-\Sigma(p_N, q_N, p_N q_N - 1) \sqcup (\bigsqcup_{c_i < 0} \Sigma(p_i, q_i, 4p_i q_i - 1))$ cobound a smooth 4-manifold that has negative definite intersection form and that satisfies $H_1(X, \mathbb{Z}/2) = 0$, contradicting Theorem 2. Therefore, Q cannot exist, and so the 3-manifolds $\Sigma(D^{r_i}(K_i))$ are independent in the $\mathbb{Z}/2$ homology cobordism group. \square

To summarize, to show that a sum $\#_{i=1}^N c_i Y_i$ of $\mathbb{Z}/2$ of homology spheres does not bound a putative $\mathbb{Z}/2$ homology ball Q , attach negative definite cobordisms either to cap off Q or to simplify its boundary. Then use gauge theoretical techniques to rule out the existence of the 4-manifold obtained and thus of the putative ball itself. Ideally, one would obtain a closed 4-manifold and then apply Donaldson's theorem since that would give the strongest possible result. In the case under consideration, sums with only positive coefficients pose no problems since double covers of Whitehead doubles bound negative definite manifolds and so can be capped-off. However, sums that involve positive coefficients pose a problem. Indeed, if we use the positive cobordism from Section 4, we end up with Seifert fibered spheres that are known not to bound a negative definite 4-manifold. Similarly, if we use the positive definite cobordism P from $-\Sigma(D^r(T_{p,q}))$ to $2S^3_{1/4}(D^{r-1}(T_{p,q}))$ described in [23], we end up with 3-manifolds that Hedden [15] shows do not bound a negative definite smooth 4-manifold since $\tau(T_{p,q}) > 0$. Also, the cobordisms constructed in Section 4 do not keep track of the index of the iteration, and so our technique cannot be used to prove independence of, for example, the family $\{D^r(T_{2,3})\}_{r \geq 1}$. Regardless, the following corollary provides a condition under which a family of iterated doubles of positive torus knots is independent. The corollary follows from Theorem 8 in the way outlined in Theorem 3.

Corollary 9. *Let $\{(p_i, q_i)\}_i$ be a sequence of relatively prime positive integers, and let r_i be a positive integer ($i = 1, 2, \dots$). If $p_i q_i (4p_i q_i - 1) < p_{i+1} q_{i+1} (p_{i+1} q_{i+1} - 1)$, then the family $\{D^{r_i}(T_{p_i, q_i})\}_{i=1}^\infty$ is independent in \mathcal{C} .*

ACKNOWLEDGMENTS

I would like to thank the organizers of the conference “Topology in dimension 3.5”, since the conference provided a favorable environment to conceive the first ideas of the present work. I would also like to thank the organizers and participants of the “Conference on 4-manifolds and knot concordance” for bearing with me as I presented an earlier and mistaken version of the main result of the present article. Special thanks go to David Krcatovich, Marco Golla, and Marco Marengon for spending enough time paying careful attention to my arguments and finally finding the mistake in my logic, and to Matt Hedden for listening to the correct proof of the theorem.

REFERENCES

- [1] S. Akbulut, Observation, CBMS conference at Santa Barbara, 1983.
- [2] Selman Akbulut, *4-manifolds*, Oxford Graduate Texts in Mathematics, vol. 25, Oxford University Press, Oxford, 2016. MR3559604
- [3] Tim D. Cochran and Robert E. Gompf, *Applications of Donaldson's theorems to classical knot concordance, homology 3-spheres and property P*, *Topology* **27** (1988), no. 4, 495–512, DOI 10.1016/0040-9383(88)90028-6. MR976591
- [4] S. K. Donaldson, *An application of gauge theory to four-dimensional topology*, *J. Differential Geom.* **18** (1983), no. 2, 279–315. MR710056
- [5] S. K. Donaldson, *Self-dual connections and the topology of smooth 4-manifolds*, *Bull. Amer. Math. Soc. (N.S.)* **8** (1983), no. 1, 81–83, DOI 10.1090/S0273-0979-1983-15090-5. MR682827
- [6] S. K. Donaldson, *The orientation of Yang-Mills moduli spaces and 4-manifold topology*, *J. Differential Geom.* **26** (1987), no. 3, 397–428. MR910015
- [7] Rob Kirby (ed.), *Problems in low-dimensional topology*, Geometric topology (Athens, GA, 1993), AMS/IP Stud. Adv. Math., vol. 2, Amer. Math. Soc., Providence, RI, 1997, pp. 35–473. MR1470751

- [8] Hisaaki Endo, *Linear independence of topologically slice knots in the smooth cobordism group*, Topology Appl. **63** (1995), no. 3, 257–262, DOI 10.1016/0166-8641(94)00062-8. MR1334309
- [9] Ronald Fintushel and Ronald J. Stern, *Instanton homology of Seifert fibred homology three spheres*, Proc. London Math. Soc. (3) **61** (1990), no. 1, 109–137, DOI 10.1112/plms/s3-61.1.109. MR1051101
- [10] Michael Hartley Freedman, *The topology of four-dimensional manifolds*, J. Differential Geom. **17** (1982), no. 3, 357–453. MR679066
- [11] Michael H. Freedman, *The disk theorem for four-dimensional manifolds*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983), PWN, Warsaw, 1984, pp. 647–663. MR804721
- [12] Michael H. Freedman and Frank Quinn, *Topology of 4-manifolds*, Princeton Mathematical Series, vol. 39, Princeton University Press, Princeton, NJ, 1990. MR1201584
- [13] Mikio Furuta, *Homology cobordism group of homology 3-spheres*, Invent. Math. **100** (1990), no. 2, 339–355, DOI 10.1007/BF01231190. MR1047138
- [14] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR1867354
- [15] Matthew Hedden, *Knot Floer homology of Whitehead doubles*, Geom. Topol. **11** (2007), 2277–2338, DOI 10.2140/gt.2007.11.2277. MR2372849
- [16] Matthew Hedden and Paul Kirk, *Instantons, concordance, and Whitehead doubling*, J. Differential Geom. **91** (2012), no. 2, 281–319. MR2971290
- [17] Jim Hoste, *A formula for Casson's invariant*, Trans. Amer. Math. Soc. **297** (1986), no. 2, 547–562, DOI 10.2307/2000539. MR854084
- [18] Charles Livingston and Paul Melvin, *Abelian invariants of satellite knots*, Geometry and topology (College Park, Md., 1983/84), Lecture Notes in Math., vol. 1167, Springer, Berlin, 1985, pp. 217–227, DOI 10.1007/BFb0075225. MR827271
- [19] Louise Moser, *Elementary surgery along a torus knot*, Pacific J. Math. **38** (1971), 737–745. MR0383406
- [20] Kunio Murasugi, *Knot theory & its applications*, translated from the 1993 Japanese original by Bohdan Kurpita, reprint of the 1996 translation [MR1391727], Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2008. MR2347576
- [21] Walter D. Neumann and Don Zagier, *A note on an invariant of Fintushel and Stern*, Geometry and topology (College Park, Md., 1983/84), Lecture Notes in Math., vol. 1167, Springer, Berlin, 1985, pp. 241–244, DOI 10.1007/BFb0075227. MR827273
- [22] Kyungbae Park, *On independence of iterated Whitehead doubles in the knot concordance group*, J. Knot Theory Ramifications **27** (2018), no. 1, 1850003, 17, DOI 10.1142/S0218216518500037. MR3749499
- [23] Juanita Pinzón-Caicedo, *Independence of satellites of torus knots in the smooth concordance group*, Geom. Topol. **21** (2017), no. 6, 3191–3211, DOI 10.2140/gt.2017.21.3191. MR3692965
- [24] Lee Rudolph, *Quasipositivity as an obstruction to sliceness*, Bull. Amer. Math. Soc. (N.S.) **29** (1993), no. 1, 51–59, DOI 10.1090/S0273-0979-1993-00397-5. MR1193540
- [25] Lee Rudolph, *An obstruction to sliceness via contact geometry and “classical” gauge theory*, Invent. Math. **119** (1995), no. 1, 155–163, DOI 10.1007/BF01245177. MR1309974
- [26] Nikolai Saveliev, *Invariants for homology 3-spheres*, Low-dimensional topology, I, Encyclopaedia of Mathematical Sciences, vol. 140, Springer-Verlag, Berlin, 2002. MR1941324
- [27] H. Seifert, *On the homology invariants of knots*, Quart. J. Math., Oxford Ser. (2) **1** (1950), 23–32, DOI 10.1093/qmath/1.1.23. MR0035436
- [28] W. Thurston, *Geometry and topology of three-manifolds*, 1978, URL <http://library.msri.org/books/gt3m/>.

DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH, NORTH CAROLINA 27695

Email address: jpinzon@ncsu.edu